

OPERATOR SPACE STRUCTURES ON $\ell^1(n)$

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ABSTRACT. We show that the complex normed linear space $\ell^1(n)$, $n > 1$, has no isometric embedding into $k \times k$ complex matrices for any $k \in \mathbb{N}$ and discuss a class of infinite dimensional operator space structures on it.

1. INTRODUCTION

In this paper, all the normed linear spaces considered are over the field of complex numbers unless specified. It is well known that there are isometric embeddings of real $\ell^1(n)$ into real $\ell^\infty(k)$ for some k and hence into the space of $k \times k$ real matrices $M_k(\mathbb{R})$. However, we prove that $\ell^1(n)$, $n > 1$, has no isometric embedding into M_k for any $k \in \mathbb{N}$. This shows that there is no operator space structure on $\ell^1(n)$, $n > 1$, which can be induced by any $k \times k$ matrices A_1, \dots, A_n . Furthermore, we study the operators space structures on $\ell^1(n)$. We recall some definitions first.

Definition 1.1. An abstract operator space is a normed linear space V together with a sequence of norms $\|\cdot\|_k$ defined on the linear space

$$M_k(V) := \{ \langle v_{ij} \rangle | v_{ij} \in V, 1 \leq i, j \leq k \}, \quad \forall k \in \mathbb{N},$$

with the understanding that $\|\cdot\|_1$ is the norm of V and the family of norms $\|\cdot\|_k$ satisfies the compatibility conditions:

1. $\|T \oplus S\|_{p+q} = \max \{ \|T\|_p, \|S\|_q \}$ and
2. $\|ASB\|_q \leq \|A\|_{op} \|S\|_p \|B\|_{op}$

for all $S \in M_q(V)$, $T \in M_p(V)$, $A \in M_{q \times p}(\mathbb{C})$ and $B \in M_{p \times q}(\mathbb{C})$.

Let $(V, \|\cdot\|_k)$ and $(W, \|\cdot\|_k)$ be two operator spaces. A linear bijection $T : V \rightarrow W$ is said to be a complete isometry if $T \otimes I_k : (M_k(V), \|\cdot\|_k) \rightarrow (M_k(W), \|\cdot\|_k)$ is an isometry for every $k \in \mathbb{N}$. Operator spaces $(V, \|\cdot\|_k)$ and $(W, \|\cdot\|_k)$ are said to be completely isometric if there is a linear complete isometry $T : V \rightarrow W$. A well known theorem of Ruan says that any operator space $(V, \|\cdot\|_k)$ can be embedded, completely isometrically, into C^* -algebra $\mathcal{B}(\mathbb{H})$ for some Hilbert space \mathbb{H} . There are two natural operator space structures on any normed linear space V , which may coincide. These are the MIN and the MAX operator space structures defined below.

Definition 1.2 (MIN). The MIN operator space structure denoted by $\text{MIN}(V)$ on a normed linear space V is obtained by the isometric embedding of V into the C^* -algebra $C((V^*)_1)$, the space of continuous functions on the unit ball $(V^*)_1$ of the dual space V^* . Thus for $\langle v_{ij} \rangle$ in $M_k(V)$, we set

$$\| \langle v_{ij} \rangle \|_{\text{MIN}} = \sup \{ \| \langle f(v_{ij}) \rangle \| : f \in (V^*)_1 \},$$

where the norm of a scalar matrix $\langle f(v_{ij}) \rangle$ is the operator norm in M_k .

The results in this paper are from the first author's thesis "The Carathéodory-Fejér Interpolation Problems and the von-Neumann Inequality" submitted to the Indian Institute of Science, Bangalore-560012.

Key words and phrases. Operator Spaces, Finite dimensional embedding, MIN structure.

The first named author was supported by the NBHM, Government of India.

The second named author was supported by CSIR, Government of India.

Definition 1.3 (Max). Let V be a normed linear space and $(v_{ij}) \in M_k(V)$. Define

$$\|(v_{ij})\|_{MAX} = \sup \{ \| (Tv_{ij}) \| : T : V \rightarrow \mathcal{B}(\mathbb{H}) \},$$

where the supremum is taken over all isometry maps T and all Hilbert spaces \mathbb{H} . This operator space structure is denoted by $MAX(V)$.

These two operator space structures are extremal in the sense that for any normed linear space V , $MIN(V)$ and $MAX(V)$ are the smallest and the largest operator space structures on V respectively. For any normed linear space V , Paulsen [Pau92] associates a constant, namely, $\alpha(V)$, which is defined as following.

$$\alpha(V) := \sup \{ \|I_V \otimes I_k\|_{(M_k(V), \|\cdot\|_{MIN}) \rightarrow (M_k(V), \|\cdot\|_{MAX})} : k \in \mathbb{N} \}.$$

The constant $\alpha(V)$ is equal to 1 if and only if V has only one operator space structure on it. There are only a few examples of normed linear spaces for which $\alpha(V)$ is known to be 1. These include $\alpha(\ell^\infty(2)) = \alpha(\ell^1(2)) = 1$. In fact, it is known (cf. [Pis03, Page 77]) that $\alpha(V) > 1$ if $\dim(V) \geq 3$.

The map $\phi : \ell^\infty(n) \rightarrow \mathcal{B}(\mathbb{C}^n)$ defined by $\phi(z_1, \dots, z_n) = \text{diag}(z_1, \dots, z_n)$, is an isometric embedding of the normed linear space $\ell^\infty(n)$ into the finite dimensional C^* -algebra $\mathcal{B}(\mathbb{C}^n)$. Clearly, this is the MIN structure of the normed linear space $\ell^\infty(n)$. We shall, however prove that there is no such finite dimensional isometric embedding for the dual space $\ell^1(n)$. Nevertheless, we shall construct, explicitly, a class of isometric infinite dimensional embeddings of $\ell^1(n)$. Unfortunately, all of these embeddings are completely isometric to the MIN structure. In the end of this paper, using these embeddings and Parrott's example in [Mis94], we construct an operator space structure on $\ell^1(3)$, which is distinct from the MIN structure.

2. $\ell^1(n)$ HAS NO ISOMETRIC EMBEDDING INTO ANY M_k

In this section, we will show that there does not exist an isometric embedding of $\ell^1(n)$, $n > 1$, into any finite dimensional matrix algebra M_k , $k \in \mathbb{N}$. Without loss of generality, we prove this for the case of $n = 2$. For the proof of the main theorem of this section, we shall need the following lemma.

Lemma 2.1. For $k \in \mathbb{N}$ and $\theta_1, \dots, \theta_k \in [0, 2\pi)$, there exists $a_1, a_2 \in \mathbb{C}$ such that

$$\max_{j=1, \dots, k} |a_1 + e^{i\theta_j} a_2| < |a_1| + |a_2|.$$

Proof. For any two non-zero complex numbers a_1, a_2 , we have

$$\max_{j=1, \dots, k} |a_1 + e^{i\theta_j} a_2| = \max_{j=1, \dots, k} ||a_1| + e^{i(\theta_j + \phi_2 - \phi_1)} |a_2||,$$

where ϕ_1 and ϕ_2 are the arguments of a_1 and a_2 respectively. Setting $\alpha_j = \theta_j + \phi_2 - \phi_1$, we have

$$\begin{aligned} \max_{j=1, \dots, k} |a_1 + e^{i\theta_j} a_2|^2 &= \max_{j=1, \dots, k} ||a_1| + e^{i\alpha_j} |a_2||^2 \\ &= \max_{j=1, \dots, k} |a_1|^2 + |a_2|^2 + 2|a_1 a_2| \cos \alpha_j|. \end{aligned}$$

Therefore

$$\max_{j=1, \dots, k} |a_1 + e^{i\theta_j} a_2| = |a_1| + |a_2|$$

if and only if $\cos \alpha_j = 1$ for some j , that is, if and only if $\alpha_j = 0$ for some j . Choose a_1 and a_2 such that $\phi_1 - \phi_2 \neq \theta_j$ for all $j = 1, \dots, k$. The existence of such a pair a_1 and a_2 proves the lemma. \square

The referee points out that the lemma is equivalent to the statement “There is no isometric embedding of $\ell^1(n)$, $n > 1$, into $\ell^\infty(k)$ for any $k \in \mathbb{N}$.” The argument below validates this equivalence.

Suppose $S : \ell^1(2) \rightarrow \ell^\infty(k)$ defined by $S(z_1, z_2) := (a_1 z_1 + b_1 z_2, \dots, a_k z_1 + b_k z_2)$ is an isometry with smallest possible $k \in \mathbb{N}$. Then, due to the minimality of k , it follows that $|a_j| = |b_j| = 1$ for $j = 1, \dots, k$. Without loss of generality, we can assume that $a_j = 1$ for $j = 1, \dots, n$. Using Lemma 2.2, we conclude that S can not be an isometry. For the converse part, we note that Lemma 2.2 is equivalent to the statement that the linear map $S : \ell^1(2) \rightarrow \ell^\infty(k)$ defined by $S(z_1, z_2) := (z_1 + e^{i\theta_1} z_2, \dots, z_1 + e^{i\theta_k} z_2)$ can not be an isometry.

Now, we prove the main theorem of this section.

Theorem 2.2. *There is no isometric embedding of $\ell^1(2)$ into M_k for any $k \in \mathbb{N}$.*

Proof. Suppose there is a k – dimensional isometric embedding ϕ of $\ell^1(2)$. Then ϕ is induced by a pair of operators $T_1, T_2 \in M_k$ of norm 1, defined by the rule, $\phi(a_1, a_2) = a_1 T_1 + a_2 T_2$. Let U_1 and U_2 in M_{2k} be the pair of unitary maps:

$$U_i := \begin{pmatrix} T_i & D_{T_i^*} \\ D_{T_i} & -T_i^* \end{pmatrix}, i = 1, 2,$$

where D_{T_i} is the positive square root of the (positive) operator $I - T_i^* T_i$. Now, we have

$$P_{\mathbb{C}^k}(a_1 U_1 + a_2 U_2)|_{\mathbb{C}^k} = a_1 T_1 + a_2 T_2.$$

(This dilating pair of unitary maps is not necessarily commuting nor is it a power dilation!) Thus $\psi : \ell^1(2) \rightarrow M_{2k}(\mathbb{C})$ defined by $\psi(a_1, a_2) = a_1 U_1 + a_2 U_2$ is also an isometry. Since norms are preserved under unitary operations, without loss of generality, we assume $U_1 = I$ and U_2 to be a diagonal unitary, say, D . Let $D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{2k}})$. Applying Lemma 2.1, we obtain complex numbers a_1 and a_2 such that

$$\max_{j=1, \dots, 2k} |a_1 + e^{i\theta_j} a_2| < |a_1| + |a_2|.$$

Hence ψ cannot be an isometry, which contradicts the hypothesis that ϕ is an isometry. \square

Remark 2.3. Let X be a finite dimensional normed linear space. Suppose X is embedded isometrically in M_k for some $k \in \mathbb{N}$, then the standard dual operator space structure on X^* need not admit an embedding in M_n for any $n \in \mathbb{N}$.

Remark 2.4. An amusing corollary to this theorem is that the two spaces $\ell^\infty(n)$ and $\ell^1(n)$ cannot be isometrically isomorphic for $n > 1$.

Remark 2.5. Prof. G. Pisier points out that Theorem 2.2 may be true if one replaces M_k by $\mathcal{K}(\mathbb{H})$, the set of all compact operators on an infinite dimensional separable Hilbert space \mathbb{H} .

3. INFINITE DIMENSIONAL EMBEDDINGS OF $\ell^1(n)$

In this section we construct operator space structure on $\ell^1(n)$ ($n \geq 3$) which is not completely isometric to MIN structure of $\ell^1(n)$.

Let \mathbb{H}_i be a Hilbert space and T_i be a contraction on \mathbb{H}_i for $i = 1, \dots, n$. Assume that the unit circle \mathbb{T} is contained in $\sigma(T_i)$, the spectrum of T_i , for $i = 1, \dots, n$. Denote

$$\tilde{T}_1 = T_1 \otimes I^{\otimes(n-1)}, \tilde{T}_2 = I \otimes T_2 \otimes I^{\otimes(n-2)}, \dots, \tilde{T}_n = I^{\otimes(n-1)} \otimes T_n$$

and $\mathbf{T} = (\tilde{T}_1, \dots, \tilde{T}_n)$.

Theorem 3.1. *Suppose the operators $\tilde{T}_1, \dots, \tilde{T}_n$ are defined as above. Then, the function*

$$f_{\mathbf{T}} : \ell^1(n) \rightarrow \mathcal{B}(\mathbb{H}_1 \otimes \dots \otimes \mathbb{H}_n)$$

defined by $f_{\mathbf{T}}(a_1, \dots, a_n) := a_1 \tilde{T}_1 + \dots + a_n \tilde{T}_n$ is an isometry.

Proof. Since $\mathbb{T} \subset \sigma(T_i)$ and T_i is a contraction for $i = 1, \dots, n$, it follows that $\mathbb{T} \subset \partial\sigma(T_i)$ for $i = 1, \dots, n$. From (cf. [Con90, Proposition 6.7, Page 210]), we have $\mathbb{T} \subset \sigma_a(T_i)$ for $i = 1, \dots, n$, where $\sigma_a(T_i)$ is the approximate point spectrum of T_i . Thus for any $i \in \{1, \dots, n\}$ and $\lambda \in \mathbb{T}$, there exists a sequence of unit vectors $(x_m^i)_{m \in \mathbb{N}}$ in \mathbb{H}_i such that

$$\|(T_i - \lambda)(x_m^i)\| \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Now, applying the Cauchy-Schwarz's inequality, we have

$$\begin{aligned} |\langle (T_i - \lambda)(x_m^i), (x_m^i) \rangle| &\leq \|(T_i - \lambda)(x_m^i)\| \|(x_m^i)\| \\ &= \|(T_i - \lambda)(x_m^i)\| \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. Hence $\langle T_i(x_m^i), (x_m^i) \rangle \rightarrow \lambda$ as $m \rightarrow \infty$. Let (a_1, \dots, a_n) be any vector in $\ell^1(n)$ such that none of its co-ordinates is zero. Let $\lambda_1 = e^{-i \arg(a_1)}, \lambda_2 = e^{-i \arg(a_2)}, \dots, \lambda_n = e^{-i \arg(a_n)}$. Now for each $i \in \{1, \dots, n\}$, we have $(x_m^i)_{m \in \mathbb{N}}$, a sequence of unit vectors from \mathbb{H}_i , such that

$$\langle T_i(x_m^i), (x_m^i) \rangle \rightarrow \lambda_i \text{ as } m \rightarrow \infty.$$

As m goes to ∞ , we have

$$\begin{aligned} &|\langle (a_1 T_1 \otimes I^{\otimes(n-1)} + \dots + a_n I^{\otimes(n-1)} \otimes T_n)(x_m^1 \otimes \dots \otimes x_m^n), (x_m^1 \otimes \dots \otimes x_m^n) \rangle| \\ &= |a_1 \langle T_1(x_m^1), (x_m^1) \rangle + \dots + a_n \langle T_n(x_m^n), (x_m^n) \rangle| \rightarrow |a_1 \lambda_1 + \dots + a_n \lambda_n| \\ &= |a_1| + \dots + |a_n| = \|(a_1, \dots, a_n)\|_1. \end{aligned}$$

Hence $\|a_1 \tilde{T}_1 + \dots + a_n \tilde{T}_n\| \geq \|(a_1, \dots, a_n)\|_1$. Also

$$\|a_1 \tilde{T}_1 + \dots + a_n \tilde{T}_n\| \leq |a_1| \|T_1\| + \dots + |a_n| \|T_n\|.$$

Hence $\|a_1 \tilde{T}_1 + \dots + a_n \tilde{T}_n\| = \|(a_1, \dots, a_n)\|_1$, proving that $f_{\mathbf{T}}$ is an isometry.

If some of the co-ordinates in the vector (a_1, \dots, a_n) are zero, the same argument, as above, remains valid after dropping those co-ordinates. \square

An adaptation of the technique involved in the proof of Theorem 3.1, also proves the following theorem.

Theorem 3.2. *For $i = 1, \dots, n$, let T_i be a contraction on a Hilbert space \mathbb{H}_i and $\mathbb{T} \subseteq \sigma(T_i)$. Denote $\tilde{T}_i = T_1 \otimes \dots \otimes T_i \otimes I_{\mathbb{H}_{i+1} \otimes \dots \otimes \mathbb{H}_n}$ and $\mathbf{T} = (\tilde{T}_1, \dots, \tilde{T}_n)$. Then, the function*

$$g_{\mathbf{T}} : \ell^1(n) \rightarrow \mathcal{B}(\mathbb{H}_1 \otimes \dots \otimes \mathbb{H}_n)$$

defined by $g_{\mathbf{T}}(a_1, \dots, a_n) := a_1 \tilde{T}_1 + \dots + a_n \tilde{T}_n$ is an isometry.

Remark 3.3. We show that all the operator spaces induced by the isometries defined in Theorem 3.1 are completely isometric to the MIN structure.

Suppose T_1, \dots, T_n are contractions on Hilbert spaces $\mathbb{H}_1, \dots, \mathbb{H}_n$ respectively with the property that $\mathbb{T} \subseteq \sigma(T_i)$ for $i = 1, \dots, n$. Denote $\tilde{T}_1 = T_1 \otimes I_{\mathbb{H}_2} \otimes \dots \otimes I_{\mathbb{H}_n}, \dots, \tilde{T}_n = I_{\mathbb{H}_1} \otimes \dots \otimes I_{\mathbb{H}_{n-1}} \otimes T_n$. Then the map $f_{\mathbf{T}}$ defined as in the Theorem 3.1 is an isometry. The dilation theorem due to Sz.-Nagy (cf. [Pau02, Theorem 1.1, Page 7]), gives unitary maps $U_j : \mathbb{K}_j \rightarrow \mathbb{K}_j$, dilating the contraction T_j , for $j = 1, \dots, n$. The operator space structure defined by the isometry $g : \ell^1(n) \rightarrow \mathcal{B}(\mathbb{K}_1 \otimes \dots \otimes \mathbb{K}_n)$, where $g(a_1, \dots, a_n) = a_1 U_1 \otimes I_{\mathbb{K}_2 \otimes \dots \otimes \mathbb{K}_n} + \dots + a_n I_{\mathbb{K}_1 \otimes \dots \otimes \mathbb{K}_{n-1}} \otimes U_n$, is no lesser than that of $f_{\mathbf{T}}$. Since U_1, \dots, U_n are unitary maps, C^* -algebra generated by $U_1 \otimes I_{\mathbb{K}_2 \otimes \dots \otimes \mathbb{K}_n}, \dots, I_{\mathbb{K}_1 \otimes \dots \otimes \mathbb{K}_{n-1}} \otimes U_n$ is commutative. From (cf. [Pis03, Proposition 1.10, Page 24]), we conclude that g is a complete isometry.

It can similarly be shown that all the operator spaces induced by the isometries defined in Theorem 3.2 are completely isometric to the MIN structure.

3.1. Operator space structures on $\ell^1(n)$ different from the MIN structure. Parrott [Par70] provides an example of a contractive homomorphism on $\mathcal{A}(\mathbb{D}^3)$ which is not completely contractive. (Here $\mathcal{A}(\mathbb{D}^3)$ is the closure, with respect to the supremum norm on \mathbb{D}^3 , of the polynomial in 3 complex variables.) Using a triple (I, U, V) (defined below) of 2×2 unitaries, it was shown in [Mis94] that examples due to Parrott may be easily thought of as examples of linear contractive maps on $\ell^1(3)$ which are not completely contractive. Indeed this realization shows that the operator space structure on $\ell^1(3)$ can not be unique. In this section, using the example from [Mis94], we give an explicit operator space structure $\|\cdot\|_{\text{os}}$ on $\ell^1(3)$, which is not completely isometric to the MIN structure

Consider the following 2×2 unitary operators:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, U := \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \text{ and } V := \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

It is clear that the map $h : \ell^1(3) \rightarrow M_2$, defined by $h(z_1, z_2, z_3) = z_1 I + z_2 U + z_3 V$, is of norm at most 1. The computations done in [Mis94] includes the following:

$$(3.1) \quad \|I \otimes I + U \otimes U + V \otimes V\| = 3$$

and

$$(3.2) \quad \sup_{z_1, z_2, z_3 \in \mathbb{D}} \|z_1 I + z_2 U + z_3 V\| < 3.$$

Choose a diagonal operator D on $\ell^2(\mathbb{Z})$ such that $\|D\| \leq 1$ and $\mathbb{T} \subset \sigma(D)$. Define

$$\tilde{T}_1 := \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}, \tilde{T}_2 := \begin{bmatrix} U & 0 \\ 0 & D \end{bmatrix}, \tilde{T}_3 := \begin{bmatrix} V & 0 \\ 0 & D \end{bmatrix}$$

and $\hat{T}_1 = \tilde{T}_1 \otimes I \otimes I$, $\hat{T}_2 = I \otimes \tilde{T}_2 \otimes I$, $\hat{T}_3 = I \otimes I \otimes \tilde{T}_3$. Let $S_1 := \hat{T}_1 \oplus I$, $S_2 := \hat{T}_2 \oplus U$, $S_3 := \hat{T}_3 \oplus V$ be operators on a Hilbert space \mathbb{K} . Define $S : \ell^1(3) \rightarrow B(\mathbb{K})$ by $S(e_1) = S_1$, $S(e_2) = S_2$, $S(e_3) = S_3$ and extend it linearly.

From Theorem 3.1, we know that the function $(z_1, z_2, z_3) \mapsto z_1 \hat{T}_1 + z_2 \hat{T}_2 + z_3 \hat{T}_3$ is an isometry and since h is of norm at most 1, it follows that the map $(z_1, z_2, z_3) \mapsto z_1 S_1 + z_2 S_2 + z_3 S_3$ is also an isometry. Consequently, there is an operator space structure os on $\ell^1(3)$ for which S is a complete isometry. Also from (3.1), we have

$$\|S_1 \otimes I + S_2 \otimes U + S_3 \otimes V\| \geq \|I \otimes I + U \otimes U + V \otimes V\| = 3.$$

Thus $\|(I, U, V)\|_{\text{os}} = 3$, as norm of (I, U, V) is at most 3 under any operator space structure on $\ell^1(3)$. On the other hand, from (3.2), we have

$$\|(I, U, V)\|_{\text{MIN}} = \sup_{z_1, z_2, z_3 \in \mathbb{D}} \|z_1 I + z_2 U + z_3 V\| < 3.$$

It follows from [Pau02, Theorem 14.1] that if there is a map $\phi : (\ell^1(3), \text{MIN}) \rightarrow (\ell^1(3), \|\cdot\|_{\text{os}})$ which is a complete isometry, then the identity $I : (\ell^1(3), \text{MIN}) \rightarrow (\ell^1(3), \|\cdot\|_{\text{os}})$ must be also a complete isometry. Therefore the operator space structure $\|\cdot\|_{\text{os}}$ is different from the MIN structure. However, although $\|(I, U, V)\|_{\text{MAX}} = 3$, we are unable to decide whether the operator space structure $\|\cdot\|_{\text{os}}$ is completely isometric to the MAX operator space structure or not.

Acknowledgement: We are very grateful to G. Misra for several fruitful discussions and suggestions. We are also thankful to the referee for many valuable suggestions.

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